Portfolio Selection with Multiple Criteria

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Contents

Overview

- Definition of multiple (conflicting) criteria objectives with respect to portfolio optimization problems.
- Concept of non-dominated solutions (Pareto efficiency).
- Solutions/points can determined by:
  1. GA/EMO: elitist non-dominated sorting algorithms (*e.g.* NSGA-II)
  2. MCDM: (classical) optimization (*e.g.* weighted-sum method)
- Applied to multi-asset portfolio with objectives for:
  1. Mean return,
  2. Total Covariance Risk,
  3. Diversification with respect to assets' risk contributions.

Multi Criteria Optimization

Problem formulation

$$\left.\begin{array}{lll} \text{minimize} & f_m(\boldsymbol{\omega}), & m = 1, 2, \ldots, M; \\ \text{subject to} & g_j(\boldsymbol{\omega}) \geq 0, & j = 1, 2, \ldots, J; \\ & h_k(\boldsymbol{\omega}) = 0, & k = 1, 2, \ldots, K; \\ & \omega_i^{(L)} \leq \omega_i \leq \omega_i^{(U)}, & i = 1, 2, \ldots, n. \end{array}\right\}$$

- Problem: M (conflicting) objective functions and n (constrained) variables.
- A solution $\hat{\boldsymbol{\omega}} \in \Omega$ is efficient (Pareto optimal or non-dominated) if there is no $\boldsymbol{\omega} \in \Omega$ such that $f_k(\boldsymbol{\omega}) \leq f_k(\hat{\boldsymbol{\omega}})$ for $k = 1, \ldots, p$ and $f_i(\boldsymbol{\omega}) < f_i(\hat{\boldsymbol{\omega}})$ for some $i \in \{1, \ldots, k\}$.

Multi Criteria Optimization

GA/EMO: Pareto efficient solutions

- Goal: find solutions which lie on Pareto-efficient front and encompass its entire range.
- Solutions can be determined by genetic NSGA-II algorithm.
- However, no guarantee in finding optimial points on (close to) frontier.
- NSGA-II consists of the following steps (a) create random population, (b) selection (non-dominant/constraint-dominant), (c) applying variation (crossover, mutation), (d) elitism (crowding distances).
- An implementation is provided in the R package **mco** (see Mersmann, 2014).

Multi Criteria Optimization

MCDM: Weighted-sum method

$$
\left.\begin{array}{llr}
\text{minimize} & \displaystyle\sum_{m=1}^{M} \lambda_m f_m(\boldsymbol{\omega}), & \text{with} \quad \lambda_m \geq 0; \\
\text{subject to} & g_j(\boldsymbol{\omega}) \geq 0, & j = 1, 2, \ldots, J; \\
& h_k(\boldsymbol{\omega}) = 0, & k = 1, 2, \ldots, K; \\
& \omega_i^{(L)} \leq \omega_i \leq \omega_i^{(U)}, & i = 1, 2, \ldots, n.
\end{array}\right\}
$$

- Scaling/normalization of objective functions is important.
- Goals and/or satisfycing levels can be included.
- Possibility of a hybrid approach, where some objectives are specified as ϵ-constraints.

Multi Criteria Optimization

Synopsis of portfolio applications

- Computation of non-dominated surface in tri-criterion problems: Hirschberger et al. (2013).
- Bi-criterion EMO problem formulation with discontinuities in search space (risk & return with (i) either zero or within bounds allocations and (ii) cardinality constraint on count of included assets): Deb et al. (2011).
- Tri-criterion problems (quad-lin-lin):
  - risk, return and& transaction costs: Steuer et al. (2005), Steuer et al. (2013).
  - risk, return and& ESG index: Utz et al. (2015).

Example: Multi-asset class portfolio

Specification

- Data set `MultiAsset` contained in **FRAPO**: month-end data from 11/2004 until 11/2011.
- MCO conisting of three objectives:
  1. mean return,
  2. volatility, and
  3. dispersion of risk contributions.
- Targeted return of 6% p.a.
- Targeted volatility of 4% p.a.

Example: Multi-asset class portfolio

R code: Initializing

```
> library(FRAPO)
> library(mco)
> ## Loading of data
> data(MultiAsset)
> Prices <- timeSeries(MultiAsset,
+                      charvec = rownames(MultiAsset))
> NAssets <- ncol(Prices)
> R <- returns(Prices, method = "discrete", percentage = TRUE)
> ## Defining parameters
> TargetRpa <- 6 ## percentage p.a.
> TargetR <- 100 * ((1 + TargetRpa / 100)^(1 / 12) - 1)
> TargetVol <- 4 ## percentage p.a.
> l <- rep(1, 3) ## goal weighting
> WeightedSum <- FALSE
> mu <- colMeans(R)
> S <- cov(R)
```

Example: Multi-asset class portfolio

R code: Multiple criteria objective and budget constraint

```
> f <- function(x){
+     y <- numeric(3)
+     y[1] <- -1.0 * l[1] * drop(crossprod(x, mu)) / TargetR
+     y[2] <- l[2] * drop(sqrt(t(x) %*% S %*% x)) *
+         sqrt(12) / TargetVol
+     y[3] <- l[3] * sum((mrc(x, S) / 100)^2)
+     if(WeightedSum){
+         return(sum(y))
+     } else {
+         return(y)
+     }
+ }
> g <- function(x){
+     c(1.01 - sum(x), sum(x) - 0.99)
+ }
```

Example: Multi-asset class portfolio

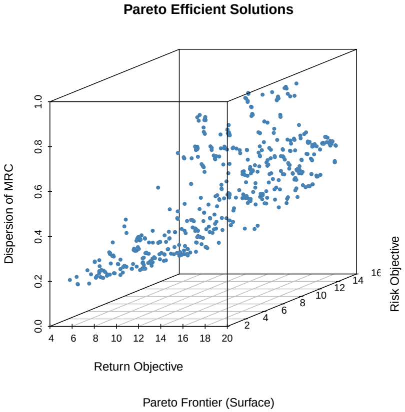
R code: Determining Pareto efficient solutions

```
> ans <- nsga2(f, NAssets, 3,
+              lower.bounds = rep(0, NAssets),
+              upper.bounds = rep(1, NAssets),
+              constraints = g, cdim = 2, popsize = 500)
> names(ans)

[1] "par"               "value"             "pareto.optimal"

> ## Preparing objective values for graphics
> mco <- data.frame(ans$value[ans$pareto.optimal, ])
> mco[, 1] <- ((1 + (-1.0 * mco[, 1] * TargetR) / 100)^12
+              - 1.0) * 100
> mco[, 2] <- mco[, 2] * TargetVol
> colnames(mco) <- c("Return", "Risk", "Diversification")
```

Example: Multi-asset class portfolio

R code: 3D scatterplot

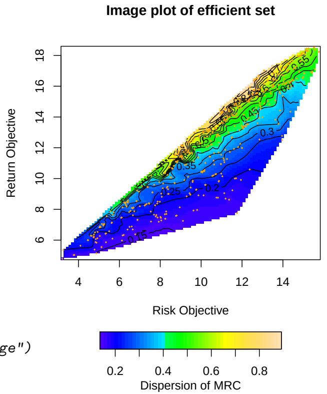
```
> library(scatterplot3d)
> scatterplot3d(mco,
+               main = "Pareto Efficient Solutions",
+               sub = "Pareto Frontier (Surface)",
+               xlab = "Return Objective",
+               ylab = "Risk Objective",
+               zlab = "Dispersion of MRC",
+               angle = 15,
+               highlight.3d = FALSE,
+               box = TRUE,
+               color = "steelblue",
+               pch = 19, type = "p",
+               cex.symbols = 0.6)
```



Example: Multi-asset class portfolio

R code: image plot

```
> library(akima)
> library(fields)
> s <- interp(mco[, 2], mco[, 1], mco[, 3],
+       xo = seq(min(mco[, 2]), max(mco[, 2]), length = 100),
+       yo = seq(min(mco[, 1]), max(mco[, 1]), length = 100),
+       duplicate = "mean"
+       )
> par(mar = c(5, 6, 5, 6))
> image.plot(s, nlevel = 50,
+            main = "Image plot of efficient set",
+            legend.lab = "Dispersion of MRC",
+            xlab = "Risk Objective",
+            ylab = "Return Objective",
+            legend.mar = 4,
+            horizontal = TRUE,
+            legend.shrink = 0.7,
+            col = topo.colors(50))
> contour(s, add = TRUE, nlevels = 20, labcex = 0.8)
> points(mco[, 2], mco[, 1], pch = 18, cex = 0.4, col = "orange")
```



Example: Multi-asset class portfolio

R code: weighting of objectives

```
grid <- expand.grid(x = seq(0.05, 0.95, by = 0.05),
                    y = seq(0.05, 0.95, by = 0.05))
grid <- grid[which(rowSums(grid) <= 1.0), ]
wobj <- as.matrix(cbind(grid, 1 - rowSums(grid)),
                  nrow = nrow(grid), ncol = 3)
W <- matrix(NA, nrow = nrow(wobj), ncol = NAssets)
WeightedSum <- TRUE
IneqA <- matrix(1, nrow = 1, ncol = NAssets)
ew <- rep(1 / NAssets, NAssets)
library(fPortfolio) ## for donlp2NLP
for(i in 1:nrow(wobj)){
    l <- c(wobj[i, ])
    W[i, ] <- donlp2NLP(start = ew, objective = f,
                 par.lower = rep(0, NAssets),
                 ineqA = IneqA, ineqA.lower = 1.0,
                 ineqA.upper = 1.0)$solution
}
```

Example: Multi-asset class portfolio

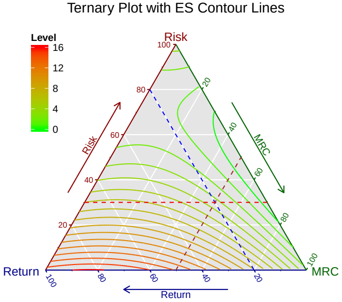
R code: weighting of objectives & ternary plot

```
> library(PerformanceAnalytics)
> library(ggtern) ## Wahrschau! version < 2.0.1
> Es95Mod <- apply(W, 1, function(x){
+     r <- timeSeries(R %*% x / 100, time(R))
+     -100 * ES(r)
+ })
> terndat <- data.frame(cbind(wobj, Es95Mod))
> colnames(terndat) <- c("x", "y", "z", "value")
> ## Theme for ternary plot
> terntheme <- function(){
+     list(theme_rgbg(),
+          theme(legend.position = c(0, 1),
+                legend.justification = c(0, 1),
+                plot.margin=unit(c(0, 2,0, 2), "cm"))
+          )
+ }
```

Example: Multi-asset class portfolio

R code: ternary plot, cont'd

```
> ## ternary plot
> ggtern(terndat, aes(x = x, y = y, z = z, value = value))
+ geom_interpolate_tern(aes(value = value, color = ..level
+                        binwidth = 1.0) +
+ terntheme() +
+ theme_hidegrid_minor() +
+ theme_showgrid_major() +
+     Lline(0.2, color = "blue", linetype = 2) + ## x
+     Tline(0.3, colour = "red2", linetype = 2) + ## y
+     Rline(0.5, color = "brown", linetype = 2) + ## z
+ scale_color_gradient(low = "green", high = "red") +
+ labs(x = "Return", y = "Risk", z = "MRC",
+      title = "Ternary Plot with ES Contour Lines",
+      color = "Level")
```



Example: Multi-asset class portfolio

R code: backtest, part I

```
> library(cccp) ## for ERC portolio
> ## backtest, extending window
> ep <- time(R)[-c(1:59)]
> bs <- length(ep)
> sp <- rep(start(R), bs)
> ## initialising object
> Wmco <- matrix(NA, nrow = bs, ncol = NAssets)
> Wmsr <- Wmdp <- Wgmv <- Werc <- Wmco
> l <- c(0.2, 0.1, 0.7) ## goal weighting
```

Example: Multi-asset class portfolio

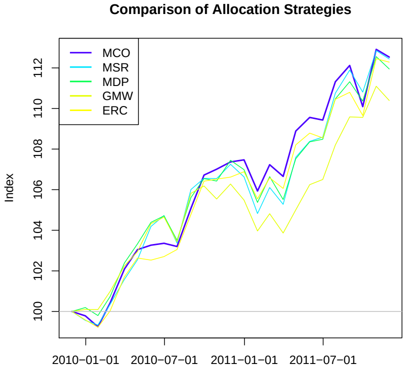
R code: backtest, part II

```
> for(i in 1:bs){
+     rdat <- window(R, start = sp[i], end = ep[i])
+     mu <- colMeans(rdat)
+     S <- cov(rdat)
+     Wmco[i, ] <- donlp2NLP(start = ew, objective = f,
+                       par.lower = rep(0, NAssets),
+                       ineqA = IneqA, ineqA.lower = 1.0,
+                       ineqA.upper = 1.0)$solution
+     ans <- tangencyPortfolio(rdat)
+     Wmsr[i, ] <- getWeights(ans)
+     ans <- PMD(rdat)
+     Wmdp[i, ] <- FRAPO::Weights(ans) / 100
+     ans <- PGMV(rdat)
+     Wgmv[i, ] <- FRAPO::Weights(ans) / 100
+     ans <- rp(ew, S, ew, optctrl = ctrl(trace = FALSE))
+     Werc[i, ] <- c(getx(ans))
+ }
```

Example: Multi-asset class portfolio

R code: backtest, part III

```
> W <- list("MCO" = Wmco, "MSR" = Wmsr, "MDP" =  Wmdp,
+           "GMV" = Wgmv, "ERC" = Werc)
> E <- lapply(W, function(x){
+     wTs <- timeSeries(x, charvec = ep)
+     wTsL1 <- lag(wTs, 1)
+     RetFac <- 1 + rowSums(R[ep, ] * wTsL1) / 100.0
+     RetFac[1] <- 100
+     timeSeries(cumprod(RetFac), charvec = ep)
+ })
> cols <- topo.colors(6)
> plot(E[[1]], lwd = 2,
+      ylab = "Index", xlab = "", col = cols[1],
+      main = "Comparison of Allocation Strategies")
> lines(E[[2]], col = cols[2])
> lines(E[[3]], col = cols[3])
> lines(E[[4]], col = cols[4])
> lines(E[[5]], col = cols[5])
> legend("topleft",
+        legend = c("MCO", "MSR", "MDP",  "GMW", "ERC"),
+        col = cols, lty = 1, lwd = 2)
> abline(h = 100, col = "gray")
```



Example: Multi-asset class portfolio

R code: backtest, part IV

```
> Rstrat <- matrix(unlist(lapply(E, Return.calculate)), ncol = 5)
> RstratTs <- na.omit(xts(Rstrat, order.by = as.Date(ep)))
> Bench <- xts(rep(0, nrow(RstratTs)), order.by = as.Date(ep)[-1])
> S1 <- as.matrix(table.AnnualizedReturns(RstratTs, Rf = Bench,
+                                         scale = 12))
> S2 <- VaR(RstratTs)
> ans <- rbind(S1, -100 * S2)
> colnames(ans) <- c("MCO", "MSR", "MDP", "GMV", "ERC")
> rownames(ans) <- c("Return (p.a.)", "StdDev. Risk (p.a.)",
+                    "Sharpe Ratio", "VaR (p.a.)")
> round(ans, 3)

                      MCO   MSR   MDP   GMV   ERC
Return (p.a.)       0.061 0.060 0.058 0.051 0.060
StdDev. Risk (p.a.) 0.038 0.039 0.037 0.034 0.034
Sharpe Ratio        1.605 1.532 1.585 1.513 1.746
VaR (p.a.)          1.301 1.380 1.308 1.211 1.017
```

Summary

- Aiding decision makers by making portfolio choices for conflicting objectives (*a posteriori analysis*).
- Allows amendment of classical portfolio optimization formulations (*e.g.* GMV, ERC, MDP and/or MSR) by additional goals.
- MCDM: For tri-criterion formulations, depiction of solutions by a fourth portfolio characteristic/measure is feasible by means of ternary plots.
- Caveat/strength of EMO: It is at the user's discretion to chose his 'optimal' allocation out of the Pareto efficient set, which might be a challenge on its own.

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